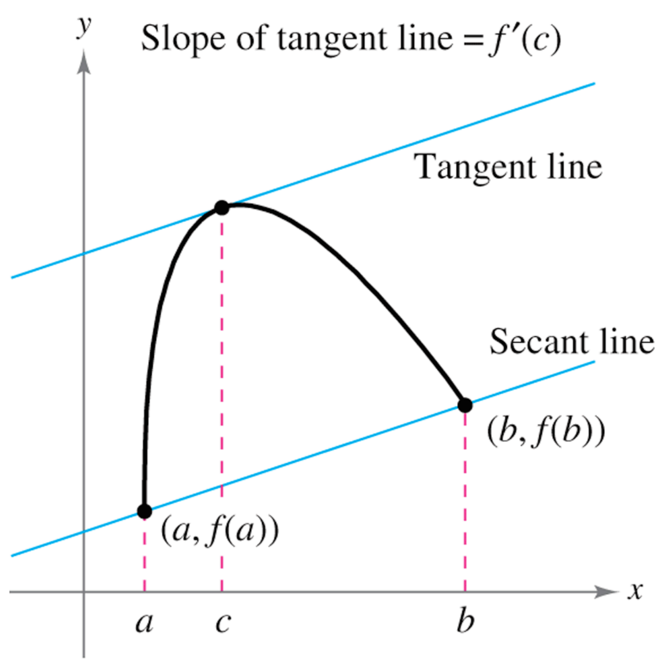
Mean Value Theorem

-If $y = f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b), then there is at least one point c in (a,b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



-Somewhere between points a and b on the differentiable curve, there is at least one tangent line parallel to the chord ab.

Example

-Show that $f(x) = x^2$ satisfies the hypothesis of the MVT on $[0,2]$. Then find a c for:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

-$f(x) = x^2$ is continuous on $[0,2]$ and differentiable on $(0,2)$

So,

$$2c = \frac{f(2) - f(0)}{2 - 0} = 2$$

$$c = 1$$

-Tangent line to $f(x) = x^2$ at $x = 0$ has slope 2 and is parallel to the chord connecting $(0,0)$ and $(2,4)$.

Rolle's Theorem

Let *f* be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If

$$f(a) = f(b)$$

then there is at least one number *c* in (a,b) such that $f'(c) = 0$.

Example

Find the two intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(x) = 0$ at some point between the x-intercepts.

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1)=0$$

So, $f(1)=f(2)=0$ and we know by Rolle's Theorem there exist at least one c on $(1,2)$ such that $f'(c)=0$.

$$f'(x)=2x-3=0$$

$f'(x)=0$ when $x=\frac{3}{2}$.

-Note that this value does fall within the interval $(1,2)$.

Example

Let $f(x)=x^4-2x^2$. Find all values of c in the interval $(-2,2)$ such that $f'(c)=0$.

-Continuous on $[-2,2]$? YES!

-Differentiable on $(-2,2)$? YES!

$f(a)=f(b)$?

$$x^4-2x^2=0$$
$$x^2(x^2-2)=0$$

$$x=0,\pm\sqrt{2}$$

$$f'(x)=4x^3-4x=0$$

$4x(x-1)(x+1)=0$

$x=0,-1,1$

Example

Let $f(x)=\sqrt{1-x^2}$, $A(-1,f(-1))$ and $B(1,f(1))$. Find a tangent to f in the interval $(-1,1)$ that is parallel to the secant AB.

-f is continuous on $[-1,1]$

$$f'(x)=\frac{-x}{\sqrt{1-x^2}}$$

is defined on $(-1,1)$

-Since $f(-1)=f(1)=0$ and the tangent we are looking for is horizontal.

-We find $f'=0$ at $x=0$ where the graph has a horizontal tangent at $y=1$.

Increasing and Decreasing Functions

-Let f be a function defined on an interval I and let x_1 and x_2 be any two points on I

f increases on I iff $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

f decreases on I iff $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Corollary-Increasing and Decreasing Functions

-Let f be continuous on $\left[a,b\right]$ and differentiable on $\left(a,b\right)$

If $f' > 0$ at each point of $\left(a,b\right)$ then f increases on $\left[a,b\right]$

If $f' < 0$ at each point of $\left(a,b\right)$ then f decreases on $\left[a,b\right]$

Example

Find where $y = x^2$ is increasing or decreasing.

-decreasing on $\left(-\infty,0\right]$ b/c $y' = 2x < 0$ on $\left(-\infty,0\right)$

-increasing on $\left[0,\infty\right)$ b/c $y' = 2x > 0$ on $\left(0,\infty\right)$

Example

Find where the function $f\left(x\right) = x^3 - 4x$ is increasing or decreasing.

-Find the derivative

$$f'\left(x\right) = 3x^2 - 4$$

-The function is increasing where $f'\left(x\right) > 0$

$$3x^2 - 4 > 0$$

$$x^2 > \frac{4}{3}$$

$x < -\sqrt{\frac{4}{3}}$ or $x > \sqrt{\frac{4}{3}}$

-The function is decreasing where $f'(x) < 0$

$3x^2 - 4 < 0$

$x^2 < \frac{4}{3}$

$-\sqrt{\frac{4}{3}} < x < \sqrt{\frac{4}{3}}$

-The function increases on $(-\infty, -1.15]$ and $[1.15, \infty)$

-Decreases on $[-1.15, 1.15]$